The problem of prescribed critical functions

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Abstract

Let (M,g) be a compact Riemannian manifold on dimension $n \geq 4$ not conformally diffeomorphic to the sphere S^n. We prove that a smooth function f on M is a critical function for a metric $\tilde{g}$ conformal to g if and only if there exists $x \in M$ such that $f(x) > 0$.

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1 Introduction

1.1 Critical functions

Let (M,g) be a compact Riemannian manifold on dimension $n \geq 3$. The Sobolev embedding $H_1^2(M)$ into $L^N(M)$ ($N = \frac{2n}{n-2}$) asserts that there exists two constants $A, B > 0$ such that, for all $u \in H_1^2(M)$,

$$\left(\int_M |u|^N dv_g \right)^{\frac{2}{N}} \leq A \int_M |\nabla u|^2 dv_g + \int_M B u^2 dv_g \qquad S(A,B)$$

Here, $H_1^2(M)$ is the set of functions $u \in L^2(M)$ such that $\nabla u \in L^2(M)$. It is well known that the best constant A in this inequality is

$$A = K(n,2)^2 = \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}$$

where ω_n stands for the volume of the standard n-dimensional sphere. As shown by Hebey and Vaugon [6], this best constant is attained. In other

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words, there exists $B > 0$ such that $S(K(n,2)^2, B)$ is true for all $u \in H_1^2(M)$. We note $B_0(g)$ the smallest constant B such this assertion is true. Clearly, $S(K(n,2)^2, B_0(g))$ holds for all $u \in H_1^2(M)$. This inequality is sharp. Moreover, we have (see the general reference [5])

$$B_0(g) \geq \max\big(\frac{n-2}{4(n-1)} K(n,2)^2 \max_M S_g, Vol_g(M)^{-\frac{2}{n}}\big) \qquad (*)$$

where S_g the scalar curvature of g. Natural questions are then:

-does there exists extremal functions in $S(K(n,2)^2, B_0(g))$? Extremal functions are nonzero functions for which $S(K(n,2)^2, B_0(g))$ is an equality.
-is it possible that $(*)$ is an equality?
These questions seem to be independent but Djadli and Druet proved in [2] that one of the following assertions must hold if $n \geq 4$:

a)$B_0(g) = \frac{n-2}{4(n-1)} K(n,2)^2 \max_M S_g$;
b)$S(K(n,2)^2, B_0(g))$ possesses extremal functions.

Then, other questions arises naturally:
-is it possible that a) is true and b) is false?
-is it possible that b) is true and a) is false?
-is it possible that a) and b) are true?
Critical functions have been introduced by Hebey and Vaugon in [7] in the purpose of answering this type of questions. The idea was to consider inequality $S(K(n,2)^2, B_0(g))$ in a metric $\tilde{g}$ conformal to g. Namely, if $\tilde{g} = u^{\frac{4}{n-2}} g$ where $u \in C^\infty(M)$, $u > 0$, then one may check that inequality $S(K(n,2)^2, B_0(g))$ is equivalent to the following one: for all $u \in H_1^2(M)$, we have

$$\left(\int_M |u|^N dv_{\tilde{g}}\right)^{\frac{2}{N}} \leq K(n,2)^2 \int_M |\nabla u|^2 dv_{\tilde{g}} + \int_M f u^2 dv_{\tilde{g}} \qquad S'(f,\tilde{g})$$

where $f \in C^\infty(M)$ satisfies

$$\Delta_g u + B_0(g) u = f u^{N-1}$$

Note that this implies that $B_0(\tilde{g}) \leq \max(f)$. It is then natural to introduce the notion of critical function. Critical functions corresponds to "best functions" in inequality above. More precisely,

Definition 1.1 (Hebey, Vaugon [7]) *We say that a smooth function f is critical for a metric g if $S'(K(n,2)^2 f, g)$ is true for all $u \in H_1^2(M)$ and if for all smooth function $f' \leq f$ with $f' \neq f$, inequality $S'(K(n,2)^2 f', g)$ is not true.*

Another way to define critical functions is the following. For any $u \in H_1^2(M) - \{0\}$, we define:

$$I_{\tilde{g},f}(u) = \frac{\int_M |\nabla u|^2_{\tilde{g}} dv_{\tilde{g}} + \int_M f u^2 dv_{\tilde{g}}}{\left(\int_M |u|^N dv_{\tilde{g}}\right)^{\frac{2}{N}}}$$

and

$$\mu_{\tilde{g},f} = \inf_{u \in H_1^2(M) - \{0\}} I_{\tilde{g},f}(u)$$

It is well known that

$$\mu_{\tilde{g},f} \leq K(n,2)^{-2}$$

We now say that

Definition 1.2 (Hebey, Vaugon [7]) *A smooth function f is*

- *subcritical for $\tilde{g}$ if $\mu_{\tilde{g},f} < K(n,2)^{-2}$;*
- *weakly critical for $\tilde{g}$ if $\mu_{\tilde{g},f} = K(n,2)^{-2}$;*
- *critical for $\tilde{g}$ if f is weakly critical with the additional property that for any smooth function h such that $h \leq f$ and $h \neq f$, h is subcritical.*

From these definitions, we get some remarks. At first, let $\tilde{g}$ be a metric conformal to g and f, h two smooth functions on M such that $h \leq f$. Then, it is clear that: for all $u \in H_1^2(M) - \{0\}$, $I_{\tilde{g},f}(u) \leq I_{\tilde{g},h}(u)$. Hence, the fact that h is weakly critical for $\tilde{g}$ implies that f is weakly critical for $\tilde{g}$ and the fact that f is subcritical for $\tilde{g}$ implies h is subcritical for $\tilde{g}$.

A second remark is that a weakly critical function f for $\tilde{g}$ satisfies:

$$f \geq \frac{n-2}{4(n-1)} S_{\tilde{g}}$$

As one can check, this can be proved by mimicking the proof of $(*)$ in [5].

Assume now that f is weakly critical for $\tilde{g}$ and that there exists $u \in H_1^2(M) - \{0\}$ such that $I_{\tilde{g},f}(u) = K(n,2)^{-2}$. Then, f is critical for $\tilde{g}$. Indeed, if $h \leq f$ and $h \neq f$, we have $I_{\tilde{g},h}(u) < I_{\tilde{g},f}(u) = K(n,2)^{-2}$ and hence, h is subcritical. A first consequence of this remark is that if $n \geq 4$, if g is a metric such that S_g is a constant function and if M is not conformally diffeomorphic to the standard sphere, then $K(n,2)^{-2} B_0(g)$ is a critical function for g. Indeed, since M is not conformally diffeomorphic to the standard sphere, it is well known $\frac{n-2}{4(n-1)} S_g$ is subcritical for g. Hence, $B_0(g) > \frac{n-2}{4(n-1)} S_g K(n,2)^2$. By Djadli and Druet's work [2], there exists $u \in H_1^2(M) - \{0\}$ such that $I_{g,K(n,2)^{-2}B_0(g)}(u) = K(n,2)^{-2}$. The remark above gives then the result.

A second consequence is that this gives a third definition of critical functions. Namely, let f be a critical function for g. Then, for any $f' \leq f$, $f' \neq f$, we have $\mu_{g,f'} < K(n,2)^{-2}$. It is well known that this implies that $\mu_{g,f'}$ is attained. In other words, there exists a minimizing solution of equation

$$\Delta_g u + f'u = u^{N-1} \qquad (**)$$

Moreover, by remark above, for any smooth function $f' \geq f$, $f' \neq f$, $\mu_{g,f'}$ is not attained and hence, equation (**) does not possess any minimizing solution. Reciprocally, let f be a smooth function which satisfies these properties. Then, clearly f is weakly critical. Hence, there exists a critical fonction $h \leq f$ (see [7]). Assume that $h \neq f$. Then, if h' is a smooth function such that $h \leq h' \leq f$, we know that equation (**) cannot have minimizing solution. This contradicts the definition of f and hence, $h = f$ and f is a critical for g. We have proven that the following definition is equivalent to the two definitions given above.

Definition 1.3 *A smooth function f is critical for a metric g if it satisfies the two following properties:*
- for any $f' \leq f$, $f' \neq f$, equation (**) has a minimizing solution;
- for any $f' \geq f$, $f' \neq f$, equation (**) does not have minimizing solutions.

A very important property of critical functions is that they have a "good" transformation law when we make a conformal change of metric. Indeed, we set $\tilde{g} = u^{\frac{4}{n-2}} g$ where $u \in C^\infty(M)$, $u > 0$. Let $h \in C^\infty(M)$ and $v \in H_1^2(M) - \{0\}$. Then, standard computations show that

$$I_{g,h}(v) = I_{\tilde{g},\tilde{h}}(u^{-1}v)$$

where $\tilde{h}$ and h are related by the following equation:

$$\Delta_g u + hu = \tilde{h} u^{\frac{n+2}{n-2}}$$

This implies that h is critical for g if and only if $\tilde{h} = \frac{\Delta_g u + hu}{u^{\frac{n+2}{n-2}}}$ is critical for $\tilde{g}$. Another way to present this result is to say: if f is a smooth function, then f is critical for $\tilde{g} = u^{\frac{4}{n-2}} g$ if and only if $f u^{\frac{4}{n-2}} - \frac{\Delta_g u}{u}$ is critical for g.

1.2 The problem

In this paper, we consider the problem of prescribed critical function: let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ not conformally diffeomorphic to the sphere S^n and f be a smooth function. Does there exist a metric $\tilde{g}$ conformal to g such that f is a critical function for $\tilde{g}$? As explained in section 1, critical functions plays an important role in the study of sharp Sobolev inequalities. Therefore, critical functions must be studied deeply to understand better these inequalities. Moreover, this problem is closely related to important geometric problems as Yamabe problem or prescribed scalar curvature problem. Namely, for two functions $\alpha, \beta \in C^\infty$, we consider the following equation:

$$\Delta_g u + \alpha u = \beta u^{N-1}$$

This type of equation is very important in geometry. For example, the Yamabe problem (see [1]) consists in finding a smooth strictly positive solution u of this equation where $\alpha = \frac{n-2}{4(n-1)} S_g$ and where β is a constant function. On what concerns the prescribed scalar curvature problem (see again [1]), we have to find a smooth strictly positive solution u where $\alpha = \frac{n-2}{4(n-1)} S_g$ and where β is given. In our problem, it follows from last paragraph that we are lead to find a critical function h for the metric g and a smooth strictly positive solution u of this equation where $\alpha = h$ and where $\beta = f$. Then, setting $\tilde{g} = u^{\frac{4}{n-2}} g$, we obtain a conformal metric to g for which f is critical. As in the prescribed scalar curvature problem, the difficulty here comes from the fact that it cannot be solved by variationnal methods. We give a complete resolution of the problem in dimension $n \geq 4$. This is object of the following result:

Main theorem *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ not conformally diffeomorphic to the sphere S^n and let f a smooth function on M. Then, there exists a metric $\tilde{g}$ conformal to g for which f is critical if and only if there exists $x \in M$ such that $f(x) > 0$.*

Obviously, the difficulty is to show that if there exists $x \in M$ such that $f(x) > 0$ then we can find a metric $\tilde{g}$ conformal to g for which f is critical. Moreover, one can find a proof much more easier that the one we give here when $f > 0$. In other words, the difficult part of the theorem corresponds to the case when f changes of sign.

One can consider the same problem if (M, g) is conformally diffeomorphic to the standard sphere or if $n = 3$. At first, let (S_n, h_0) be the standard sphere of dimension n and g be a metric conformal to h_0. Then the only critical function for g is $\frac{n-2}{4(n-1)} S_g$. Hence, the problem is equivalent to the problem of prescribed scalar curvature. If now $n = 3$, then critical functions do not have the same properties than in dimension upper than four. For example, theorem 2.1 below is false when $n = 3$ (see [7] and [3]). The 3-dimensional case seems to be interesting but the methods used here are not adapted to this case.

2 Proof of main theorem

In this section, (M, g) is a compact Riemannian manifold of dimension $n \geq 4$ not conformally diffeomorphic to the sphere S^n and f is a smooth function on M. In addition, up to making a conformal change of metric, one may assume that S_g is a constant function and up to multiplying g by a constant, we can also assume that

$$\alpha_0 - \max_M(f) \geq \frac{n-2}{4(n-1)} S_g \tag{2.1}$$

where $\alpha_0 = K(n,2)^{-2} B_0(g)$.

2.1 A preliminary result

For the proof of main theorem, we will need the following result:

Theorem 2.1 *Let $(h_m)_m$ be a sequence of smooth functions on M which converges uniformly to a smooth function h. We assume that for all m, h_m is subcritical for g and that h is weakly critical. Moreover, we assume that*

$$h > \frac{n-2}{4(n-1)} S_g$$

Then, h is critical.

The proof follows very closely the proof of Druet and Djadli's theorem in [2]. In addition, the reader may refer to [7] sections 2 and 3 for a sketch of proof of this theorem as stated here.

2.2 Proof of main theorem

At first, if f is a critical function for any metric $\tilde{g}$ (not necessarily conformal to g), then, there exists $y \in M$ such that $f(y) > 0$. Indeed, coming back to the notations of section 1, if $f \leq 0$, we have $I_{\tilde{g},f}(1) \leq 0 < K(n,2)^{-2}$ and hence, f cannot be critical for $\tilde{g}$. Therefore, we assume that there exists $y \in M$ such that $f(y) > 0$ and we have to show that we can find a metric $\tilde{g}$ conformal to g for which f is critical. We set

$$\mathcal{F} : \left| \begin{array}{ccc} \Omega & \rightarrow & C^\infty(M) \\ u & \mapsto & f u^{\frac{4}{n-2}} - \frac{\Delta_g u}{u} \end{array} \right.$$

where

$$\Omega = \{u \in C^\infty(M) | u > 0\}$$

Let $u \in \Omega$. By paragraph 1.1, f is critical for $\tilde{g} = u^{\frac{4}{n-2}} g$ if and only if $\mathcal{F}(u)$ is critical for g. In the following, we say weakly critical, subcritical and critical and we omit to say "for g". Coming back to the notations of section 1, we set, for any smooth function h and $u \in H_1^2(M)$:

$$I_h(u) = I_{g,h}(u) = \frac{\int_M |\nabla u|_g^2 dv_g + \int_M h u^2 dv_g}{\left(\int_M |u|^N dv_g\right)^{\frac{2}{N}}}$$

and

$$\mu_h = \mu_{g,h} = \inf_{H_1^2(M) - \{0\}} I_h$$

With these notations, we want to find $u \in \Omega$ such that $\mu_{\mathcal{F}(u)} = K(n,2)^{-2}$ and for all $h \leq \mathcal{F}(u)$, $h \neq \mathcal{F}(u)$, $\mu_h < K(n,2)^{-2}$. A natural idea to prove the main theorem is then the following: we find $u, v \in \Omega$ such that $\mathcal{F}(u)$ is subcritical and $\mathcal{F}(v)$ is weakly critical. Then, we take a continuous path $(u_t)_t \subset \Omega$ for $t \in [0,1]$ such that $u_0 = u$ and $u_1 = v$. We define

$$t_0 = \inf\{t > 0 \text{ such that } \mathcal{F}(u_t) \text{ is weakly critical }\}$$

The idea is to apply theorem 2.1 with $h = \mathcal{F}(u_{t_0})$ and $h_m = \mathcal{F}(u_{t_m})$ where $t_m = t_0 - \frac{1}{m}$. The difficulty is that we need the additional assumption that $\mathcal{F}(u_{t_0}) > \frac{n-2}{4(n-1)} S_g$. The linear transformation $u_t = tu + (1-t)v$ does not work in general. Hence, we must be very careful with the choice of u, v and u_t. In fact, we show that there exists $u \in \Omega$ and $s > 1$ such that $\mathcal{F}(u)$ is subcritical and such that $\mathcal{F}(u^s)$ is weakly critical. The method described above then works with $u_t = \mathcal{F}(u^{ts})$. We strongly use in the whole proof concentration phenomenoms. In the special case where $f > 0$ then, one can find a shorter proof than the one we give here.

Let us start the proof now. For all $u \in H_1^2(M)$, $t > 0$, and $q \in]2, N]$, we set

$$J_{q,t}(u) = \frac{\int_M |\nabla u|_g^2 dv_g + t \int_M u^2 dv_g}{\left(\int_M f |u|^q dv_g\right)^{\frac{2}{q}}}$$

We define, for $q \in]2, N]$

$$\mu_{q,t} = \inf_{\mathcal{H}_q} J_{q,t}(u)$$

where

$$\mathcal{H}_q = \{u \in H_1^2(M) | \int_M f |u|^q dv_g > 0\}$$

Obviously, since there exists $y \in M$ such that $f(y) > 0$ and since f is continuous, the set $\mathcal{H}_q$ is not empty. It is well known that for all $t > 0$

$$\mu_{N,t} \leq K(n,2)^{-2} (\max_M f)^{\frac{2}{N}} \tag{2.2}$$

and that for all $u \in \mathcal{H}_q$, we have $J_t(u) = J_t(|u|)$. Hence, we can replace $\mathcal{H}_q$ by

$$\mathcal{H}_q = \{u \in C^\infty(M) | u > 0 \text{ and } \int_M f u^q dv_g > 0\}$$

We now define:

$$\Omega_{q,t} = \{u \in \mathcal{H}_q | J_{q,t}(u) = \mu_{q,t} \text{ and } \int_M f u^q dv_g = \mu_{q,t}^{\frac{q}{q-2}}\}$$

The value $\mu_{q,t}^{\frac{q}{q-2}}$ is chosen to obtain equation $E(q,t)$ below. Note that, for all $t > 0$ and $q \in]2, N[$, $\mu_{q,t} > 0$. By standard elliptic theory, we know that for all $q < N$ and all $t > 0$,

$$\Omega_{q,t} \neq \emptyset$$

Note that if t is large (for example $t > \alpha_0 = B_0(g)K(n,2)^{-2}$), $\Omega_{N,t} = \emptyset$. Indeed, let $t > \alpha_0$ and assume that there exists $u \in \Omega_{N,t}$. Then, by (2.2),

$$(\max_M f)^{-\frac{2}{N}} \frac{\int_M |\nabla u|^2 dv_g + \alpha_0 \int_M u^2}{\left(\int_M u^N dv_g\right)^{\frac{2}{N}}} < J_{N,t}(u)$$

$$= \mu_{N,t} \leq K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}}$$

This contradicts the fact that, by definition α_0 is weakly critical. Another remark is the following: if $u \in \Omega_{q,t}$ with $q \in]2, N]$ and $t > 0$, then writing the Euler equation of u, we get that u satisfies

$$\Delta_g u + tu = f u^{q-1} \qquad E(q,t)$$

We first prove that:

Step 1 *Let $0 < t < \alpha_0$ where $\alpha_0 = B_0(g)K(n,2)^{-2}$ is the lowest weakly critical constant function. Then, there exists $q_0 < N$ such that for all $q \in [q_0, N[$, and all $u \in \Omega_{q,t}$, $\mathcal{F}(u)$ is subcritical for g.*

We proceed by contradiction. We assume that there exits a sequence (q_i) of real numbers and a sequence (u_i) of functions belonging to $\Omega_{q_i,t}$ such that
- $\lim_i q_i = N$
- $q_i < N$ for all i
- $\mathcal{F}(u_i)$ is weakly critical.

Clearly, (u_i) is bounded in $H_1^2(M)$. Hence, by standard arguments (see [1] or [4]), there exists $u \in H_1^2(M)$ such that, up to a subsequence, $u_i \to u$ weakly in $H_1^2(M)$, strongly in $L^2(M)$, strongly in $L^{N-2}(M)$ and almost everywhere.

First, we assume that $u \not\equiv 0$. Then, by elliptic theory, $u \in \mathcal{H}_N$ and up to subsequence, we may assume that

$$u_i \to u \text{ in } C^2(M)$$

Therefore, the sequence $(\mathcal{F}(u_i))$ converges uniformly to $\mathcal{F}(u)$. Since $\mathcal{F}(u_i)$ is weakly critical, then $\mathcal{F}(u)$ is weakly critical too. Moreover, $u_i \in \Omega_{q_i,t}$ and hence satisfies equation $E(q_i, t)$. This gives

$$\mathcal{F}(u_i) = t + f(u_i^{N-2} - u_i^{q_i-2}) \tag{2.3}$$

Passing to the limit in i, we get that

$$\Delta_g u + tu = fu^{N-1}$$

and

$$\mathcal{F}(u) = t$$

Therefore, we have proven that the constant function t is weakly critical for g. This is impossible since $t < \alpha_0$ and since α_0 is the smallest weakly critical constant function.

We now deal with the case where $u \equiv 0$. Since $t < \alpha_0$, the constant function t is subcritical. Hence, there exists a positive function $\phi \in C^\infty(M)$ such that

$$I_t(\phi) = \frac{\int_M |\nabla \phi|_g^2 dv_g + t \int_M \phi^2 dv_g}{\left(\int_M |\phi|^N dv_g\right)^{\frac{2}{N}}} < K(n,2)^{-2} \tag{2.4}$$

Plugging the test function ϕ into $I_{\mathcal{F}(u_i)}$, we get by (2.3) that

$$I_{\mathcal{F}(u_i)}(\phi) = I_t(\phi) + \frac{\int_M f(u_i^{N-2} - u_i^{q_i-2})\phi^2 dv_g}{\left(\int_M |\phi|^N dv_g\right)^{\frac{2}{N}}}$$

By strong convergence of (u_i) to 0 in $L^{N-2}(M)$ and since $q_i - 2 \le N - 2$, we get that

$$\lim_i \int_M f u_i^{N-2} \phi^2 dv_g = \lim_i \int_M f u_i^{q_i-2} \phi^2 dv_g = 0$$

It follows that

$$\lim_i I_{\mathcal{F}(u_i)}(\phi) = I_t(\phi) < K(n,2)^{-2}$$

That contradicts the fact that $\mathcal{F}(u_i)$ is weakly critical. This ends the proof of step 1.

By step 1, one can find two sequences (q_i), (t_i) of real numbers such that
1) $2 < q_i < N$ and $\lim_i q_i = N$;
2) $t_i > 0$ and $\lim_i t_i = \alpha_0 = K(n,2)^{-2} B_0(g)$
and a sequence (v_i) of functions belonging to Ω_{q_i,t_i} with the additionnal property that $\mathcal{F}(v_i)$ is subcritical. Clearly, proceeding as in step 1, one can find $v \in H_1^2(M)$ such that, up to a subsequence, $v_i \to v$ weakly in $H_1^2(M)$, strongly in $L^2(M)$, strongly in $L^{N-2}(M)$ and almost everywhere. We set

$$J_i = J_{q_i,t_i} \text{ and } \mu_i = \mu_{q_i,t_i}$$

We prove that

Step 2 *We can assume $v \equiv 0$*

Otherwise, as in step 1,

$$v_i \to v \text{ in } C^2(M)$$

Mooreover, $v_i \in \Omega_{q_i,t_i}$. Hence, v_i satisfies equation $E(q_i,t_i)$ and we have

$$\mathcal{F}(v_i) = t_i + f(v_i^{N-2} - v_i^{q_i-2}) \tag{2.5}$$

Passing to the limit in i, we get

$$\mathcal{F}(v) = \alpha_0$$

Moreover, by maximum principle and regularity theorem, $v \in C^\infty(M)$ and $v > 0$. The construction of $\mathcal{F}$ is such that f is critical for $\tilde{g} = v^{\frac{4}{N-2}} g$ if and only if $\mathcal{F}(v)$ is critical for g. This is the case here because α_0 is critical for g. Then, the theorem is proved. Thus, in the following, we may assume that $v \equiv 0$. This proves step 2.

We now assume that $v \equiv 0$ and we prove that

Step 3 *We have :*

$$\lim_i \mu_i = K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} \textit{ and } \lim_i \int_M v_i^{q_i} dv_g = K(n,2)^{-n} (\max_M f)^{-\frac{n}{2}}$$

As easily seen, $\liminf_i \mu_i > 0$. We have, using Hölder inequality:

$$\mu_i = J_i(v_i) = \frac{\int_M |\nabla v_i|_g^2 dv_g + t_i \int_M v_i^2 dv_g}{\left(\int_M f {v_i}^{q_i} dv_g\right)^{\frac{2}{q_i}}}$$

$$\geq \frac{\int_M |\nabla v_i|_g^2 dv_g + \alpha_0 \int_M v_i^2 dv_g}{(\max_M f)^{\frac{2}{q_i}} \left(\int_M {v_i}^N dv_g\right)^{\frac{2}{N}} Vol(M)^{1-\frac{q_i}{N}}} + (t_i - \alpha_0) \frac{\int_M v_i^2 dv_g}{\left(\int_M f {v_i}^{q_i} dv_g\right)^{\frac{2}{q_i}}}$$

Since $\lim_i t_i = \alpha_0$, since $v_i \to 0$ in $L^2(M)$ and since

$$\liminf_i \int_M f v_i^{q_i} dv_g = \liminf_i \mu_i^{\frac{q_i}{q_i-2}} > 0$$

we have

$$\lim_i (t_i - \alpha_0) \frac{\int_M v_i^2 dv_g}{\left(\int_M f v_i{}^{q_i} dv_g\right)^{\frac{2}{q_i}}} = 0$$

Moreover,

$$\frac{\int_M |\nabla v_i|_g^2 dv_g + \alpha_0 \int_M v_i^2 dv_g}{\left(\int_M v_i{}^{q_i} dv_g\right)^{\frac{2}{q_i}}} = I_{\alpha_0}(v_i) \geq K(n,2)^{-2}$$

because α_0 is weakly critical. We obtain that

$$\liminf_i \mu_i \geq K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}}$$

Now, by (2.2), we can find $w \in C^\infty(M)$ such that

$$J_{N,\alpha_0}(w) \leq K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} + \epsilon$$

where $\epsilon > 0$ is as small as wanted. We have

$$\limsup_i J_i(w) = J_{N,\alpha_0}(w) \leq K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} + \epsilon$$

This proves that

$$\lim_i \mu_i = K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} \tag{2.6}$$

Now, we multiply $E(q_i, t_i)$ by v_i and we integrate over M. We get:

$$\int_M |\nabla v_i|_g^2 dv_g + t_i \int_M v_i^2 = \int_M f v_i^{q_i} dv_g \tag{2.7}$$

We recall that $\int_M f v_i^{q_i} dv_g = \mu_i^{\frac{q_i}{q_i-2}}$. Hence, with Hölder inequality:

$$\int_M f v_i^{q_i} dv_g = \mu_i \left(\int_M f v_i^{q_i} dv_g \right)^{\frac{2}{q_i}}$$

$$\leq \mu_i (\max_M f)^{\frac{2}{q_i}} \left(\int_M v_i^{q_i} dv_g \right)^{\frac{2}{q_i}} \leq \mu_i (\max_M f)^{\frac{2}{q_i}} \left(\int_M v_i^N dv_g \right)^{\frac{2}{N}} Vol(M)^{1-\frac{q_i}{N}} \tag{2.8}$$

Using inequality $S(K(n,2)^2, B_0(g))$ (see introduction), we obtain that

$$\int_M f v_i^{q_i} dv_g$$

$$\leq \mu_i (\max_M f)^{\frac{2}{q_i}} Vol(M)^{1-\frac{q_i}{N}} \left(K(n,2)^2 \int_M |\nabla v_i|_g^2 dv_g + B_0(g) \int_M v_i^2 dv_g \right)$$

Together with (2.7) and (2.8), we get

$$\int_M |\nabla v_i|_g^2 dv_g + t_i \int_M v_i^2 \leq \mu_i (\max_M f)^{\frac{2}{q_i}} \left(\int_M v_i^{q_i} dv_g \right)^{\frac{2}{q_i}}$$

$$\leq \mu_i (\max_M f)^{\frac{2}{q_i}} Vol(M)^{1-\frac{q_i}{N}} \left(K(n,2)^2 \int_M |\nabla v_i|_g^2 dv_g + B_0(g) \int_M v_i^2 dv_g \right) \quad (2.9)$$

Now, we have $J_i(v_i) = \mu_i$ and $\lim_i \| v_i \|_2 = 0$. Since $\int_M f v_i^{q_i} dv_g = \mu_i^{\frac{q_i}{q_i - 2}}$, we get from (2.6) that

$$\lim_i \int_M |\nabla v_i|_g^2 dv_g = \lim_i \mu_i^{\frac{n}{2}} = \left(K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} \right)^{\frac{n}{2}}$$

Taking the limit in i in both sides of inequality (2.9) and using (2.6),

$$\left(K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} \right)^{\frac{n}{2}} \leq K(n,2)^{-2} \left(\lim_i \int_M v_i^{q_i} dv_g \right)^{\frac{2}{N}}$$

$$\leq \left(K(n,2)^{-2} (\max_M f)^{-\frac{2}{N}} \right)^{\frac{n}{2}}$$

The step then follows immediatly.

Let now $x \in M$ be given. Following usual terminology, we say that x is a concentration point if for all $r > 0$,

$$\limsup_i \int_{B_x(r)} v_i^{q_i} dv_g > 0$$

where $B_x(r)$ stands for the geodesic ball of center x and radius r.

Step 4 *Up to a subsequence, (v_i) possesses exactly one concentration point x_0. Moreover, x_o is a point where f is maximum. If $\bar{\omega} \subset\subset M - \{x_0\}$ where ω is an open subset of M, then (v_i) tends uniformly to 0 with i on $\bar{\omega}$.*

By step 3 and since M is compact, it is easy to prove the existence of at least one point of concentration. We now let $x \in M$ and $r > 0$, a small positive number. Let also $\eta \in C^\infty(M)$ a cut-off function supported in $B_x(r)$, such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_x(\frac{r}{2})$. We recall that v_i satisfies equation $E(q_i, t_i)$. We multiply $E(q_i, t_i)$ by $\eta^2 v_i^k$ for $k > 1$ and integrate over M. We get:

$$\int_M \eta^2 v_i^k \Delta_g v_i dv_g + t \int_M v_i^{k+1} \eta^2 dv_g = \int_M f \eta^2 v_i^{k+q_i-1} dv_g \tag{2.10}$$

Integrating by parts, we get

$$\int_M |\nabla \eta v_i^{\frac{k+1}{2}}|_g^2 dv_g = \frac{(k+1)^2}{4k} \int_M \eta^2 v_i^k \Delta_g v_i dv_g$$

$$+ \frac{k+1}{2k} \int_M \left(|\nabla \eta|_g^2 + \frac{k-1}{k+1} \eta \Delta_g \eta \right) v_i^{k+1} dv_g$$

Together with (2.10), this gives

$$\int_M |\nabla \eta v_i^{\frac{k+1}{2}}|_g^2 dv_g \le \frac{(k+1)^2}{4k} \int_M f \eta^2 v_i^{k+q_i-1} dv_g$$

$$+ \frac{k+1}{2k} \int_M \left(|\nabla \eta|_g^2 + \frac{k-1}{k+1} \eta \Delta_g \eta \right) v_i^{k+1} dv_g$$

By Hölder inequality, we have

$$\int_M |\nabla \eta v_i^{\frac{k+1}{2}}|_g^2 dv_g \le \frac{(k+1)^2}{4k} \max_M (f) \left(\int_M (\eta v_i^{\frac{k+1}{2}})^{q_i} dv_g \right)^{\frac{2}{q_i}} \left(\int_{B_x(r)} v_i^{q_i} dv_g \right)^{\frac{q_i-2}{q_i}}$$

$$+ \frac{k+1}{2k} \int_M \left(|\nabla \eta|_g^2 + \frac{k-1}{k+1} \eta \Delta_g \eta \right) v_i^{k+1} dv_g \tag{2.11}$$

From inequality $S(K(n,2)^2, B_0(g))$ and again Hölder inequality, we get:

$$\int_M |\nabla \eta v_i^{\frac{k+1}{2}}|_g^2 dv_g \ge K(n,2)^{-2} \left(\int_M (\eta v_i^{\frac{k+1}{2}})^N dv_g \right)^{\frac{2}{N}} - \alpha_0 \int_M v_i^2 dv_g$$

$$\ge K(n,2)^{-2} Vol(M)^{\frac{q_i}{N}-1} \left(\int_M (\eta v_i^{\frac{k+1}{2}})^{q_i} dv_g \right)^{\frac{2}{q_i}} - \alpha_0 \int_M \eta^2 v_i^{k+1} dv_g$$

Together with (2.11), we are lead to

$$\left(\int_M (\eta v_i^{\frac{k+1}{2}})^{q_i} \right)^{\frac{2}{q_i}} \left(K(n,2)^{-2} Vol(M)^{\frac{q_i}{N}-1} - \frac{(k+1)^2}{4k} \max_{B_x(r)} (f) \left(\int_{B_x(r)} v_i^{q_i} dv_g \right)^{\frac{q_i-2}{q_i}} \right)$$

$$\leq C \int_M v_i^{k+1} dv_g \tag{2.12}$$

where $C > 0$ is a constant which does not depend on i. If x is a concentration point, then

$$\liminf_i \int_{B_x(r)} v_i^{q_i} dv_g > 0$$

Moreover, by step 3,

$$\liminf_i \int_{B_x(r)} v_i^{q_i} dv_g \leq K(n,2)^{-n} (\max_M f)^{-\frac{n}{2}} \tag{2.13}$$

Assume that this inequality is strict. Then, if k is sufficiently close to 1, we have

$$\liminf_i \left(K(n,2)^{-2} Vol(M)^{\frac{q_i}{N}-1} - \frac{(k+1)^2}{4k} \max_M (f) \left(\int_{B_x(r)} v_i^{q_i} dv_g \right)^{\frac{q_i-2}{q_i}} \right) > 0$$

Coming back to (2.12), we get the existence of $C > 0$ independent of i such that

$$\left(\int_M (\eta v_i^{\frac{k+1}{2}})^{q_i} \right)^{\frac{2}{q_i}} \leq C \int_M v_i^{k+1} dv_g \tag{2.14}$$

The right hand side of (2.14) goes to 0 with i. By Hölder inequality, we would get that

$$\lim_i \int_{B_x(\frac{r}{2})} v_i^{q_i} dv_g \leq \lim_i \int_{B_x(\frac{r}{2})} v_i^{\frac{k+1}{2} q_i} dv_g = 0$$

This is impossible since x is a concentration point. It follows that (2.13) is a equality and hence, there exists one and only one concentration point x_0. Moreover, if $\max_{B_x(r)}(f) < \max_M(f)$ (with $x = x_0$), we get (2.14) in the same way. Hence, the concentration point x_0 is such that $\max_M(f) = f(x_0)$.
Now, let $\bar{\omega} \subset\subset M - \{x_0\}$ where ω is an open set of M. Let $0 < r < dist_g(\omega, x_0)$ and a finite set (x_j) of points of ω such that

$$\omega \subset \cup_j B_{x_j}(r)$$

Doing the same with $x = x_j$, this leads to the existence of $C > 0$ such that

$$\int_\omega v_i^{\frac{k+1}{2} q_i} dv_g \leq C \int_M v_i^{k+1} dv_g$$

Since $\frac{k+1}{2} q_i > N + \epsilon$ where $\epsilon > 0$ is small, a simple application of Moser's iterative scheme proves the step.

We now let $s > 1$ be a large real number. We claim that

Step 5 *For i large enough, the function $\mathcal{F}(v_i^s)$ is weakly critical for g. Moreover, for all $t \in [1,s]$, $\mathcal{F}(v_i^t) > \frac{n-1}{4(n-2)} S_g$.*

It is sufficient to prove that for i large enough, $\mathcal{F}(v_i^s) \geq \alpha_0$. An easy computation gives

$$\Delta_g(v_i^s) = s v_i^{s-1} \Delta_g v_i - s(s-1) v_i^{s-2} |\nabla v_i|_g^2 \leq s v_i^{s-1} \Delta_g v_i$$

Since v_i satisfies $E(q_i, t_i)$, it follows that

$$\mathcal{F}(v_i^s) \geq s t_i + f(v_i^{\frac{4}{n-2}s} - s v_i^{q_i - 2}) \tag{2.15}$$

By step (4), we know that v_i converges uniformly to 0 on $\{f \leq 0\}$. Since $\frac{4}{n-2}s > q_i - 2$, we get that, on $\{f \leq 0\}$ and for i large enough

$$\mathcal{F}(v_i^s) \geq s t_i > \alpha_0 \tag{2.16}$$

Now, we set for $x \geq 0$,

$$\beta(x) = x^{\frac{4}{n-2}s} - s x^{q_i - 2} = x^{q_i - 2}(x^{\frac{4}{n-2}s - q_i + 2} - s)$$

The minimum of β on $[0, +\infty[$ is attained for

$$x_i = \left(\frac{(n-2)(q_i - 2)}{4} \right)^{\frac{1}{\frac{4}{n-2}s - q_i + 2}}$$

Moreover, $x_i \leq 1$ because $q_i \leq N$. Hence, $|x^i|^{q_i - 2} \leq 1$. Hence,

$$\beta(x_i) \geq x_i^{\frac{4}{n-2}s - q_i + 2} - s = \frac{(n-2)(q_i - 2)}{4} - s \geq -s$$

We get from (2.15) that

$$\mathcal{F}(v_i^s) \geq s(t_i - \max_M(f))$$

on $\{f \geq 0\}$. By (2.1), $\alpha_0 > \max_M(f)$. Since $\lim_i t_i = \alpha_0$, we obtain that

$$\mathcal{F}(v_i^s) \geq \alpha_0$$

on $\{f \geq 0\}$ if s is chosen large enough. By (2.16), this inequality is true on all M. This proves that $\mathcal{F}(v_i^s)$ is weakly critical.
Now let $t \in [1,s]$. In the same way, we obtain that if i is large enough,

$$\mathcal{F}(v_i^t) \geq t(t_i - \max_M(f)) \geq (t_i - \max_M(f))$$

By (2.1), $\mathcal{F}(v_i^t) > \frac{n-1}{4(n-2)} S_g$. This proves the step.

Step 6 *Conclusion*

We let i and $s > 1$ such that $\mathcal{F}(v_i)$ is subcritical, $\mathcal{F}(v_i^s)$ is weakly critical and for all $t \in [1, s]$, $\mathcal{F}(v_i^t) > \frac{n-1}{4(n-2)} S_g$. We set $v = v_i$. We also define

$$s_0 = \inf\{t > 1 | \mathcal{F}(v^t) \text{ is weakly critical } \}$$

It is clear that $\mathcal{F}(v^{s_0})$ is weakly critical. Now, let (t_m) a sequence of real numbers such that $1 < t_m < s_0$ and $\lim_m t_m = s_0$. We apply theorem 2.1 with $h_m = \mathcal{F}(v^{t_m})$ and $h = \mathcal{F}(v^{s_0})$. It follows that $\mathcal{F}(v^{s_0})$ is critical for g and hence that f is critical for $\tilde{g} = v^{\frac{4}{n-2} s_0} g$. This ends the proof of main theorem.

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